

A Mathematical Introduction to Logic

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Chapter R

Reference

R.1 ¶ Construction Sequence

A **construction sequence** is a ¶ Finite Sequence $\langle \epsilon_1, \dots, \epsilon_n \rangle$ of ¶ Expressions such that for each $i \leq n$ we have at least one of

ϵ_i is a sentence symbol

$\epsilon_i = \mathcal{E}_{\neg}(\epsilon_j)$ for some $j < i$

$\epsilon_i = \mathcal{E}_{\square}(\epsilon_j, \epsilon_k)$ for some $j < i, k < i$

where \square is one of the binary connectives $\wedge, \vee, \Rightarrow, \Leftrightarrow$.

R.2 ¶ Expression

An **expression** is a ¶ Finite Sequence of symbols.

R.3 ¶ Finite Sequence

S is a **finite sequence** (or **string**) of members of set A if and only if, for some positive integer n , we have $S = \langle x_1, \dots, x_n \rangle$, where each $x_i \in A$.

R.4 ¶ Formula-Building Operations

The **formula-building operations** (on expressions) are defined by the equations:

$$\begin{aligned}\mathcal{E}_-(\alpha) &= (\neg\alpha) \\ \mathcal{E}_\wedge(\alpha, \beta) &= (\alpha \wedge \beta) \\ \mathcal{E}_\vee(\alpha, \beta) &= (\alpha \vee \beta) \\ \mathcal{E}_\Rightarrow(\alpha, \beta) &= (\alpha \Rightarrow \beta) \\ \mathcal{E}_\Leftrightarrow(\alpha, \beta) &= (\alpha \Leftrightarrow \beta)\end{aligned}$$

🔗 - Enderton.Logic.Chapter_1.Wff

∀ - Not

∀ - And

∀ - Or

∀ - Iff

R.5 ¶ n -tuple

An n -**tuple** is recursively defined as

$$\langle x_1, \dots, x_{n+1} \rangle = \langle \langle x_1, \dots, x_n \rangle, x_{n+1} \rangle$$

for $n > 1$. We also define $\langle x \rangle = x$.

∀ - Prod

R.6 ¶ Tautological Implication

Consider a set Σ of ¶ Well-Formed Formulas and another wff τ . Σ **tautologically implies** τ (written $\Sigma \models \tau$) if and only if every ¶ Truth Assignment for the sentence symbols in Σ and τ that satisfies every member of Σ also satisfies τ . If Σ is singleton $\{\sigma\}$, then we write " $\sigma \models \tau$ " in place of " $\{\sigma\} \models \tau$."

If both $\sigma \models \tau$ and $\tau \models \sigma$, then σ and τ are said to be **tautologically equivalent** (written $\sigma \models \Leftrightarrow \tau$).

R.7 ¶ Truth Assignment

A **truth assignment** v for a set \mathcal{S} of sentence symbols is a function

$$v: \mathcal{S} \rightarrow \{F, T\}$$

assigning either T or F to each symbol in \mathcal{S} .



Let $\bar{\mathcal{S}}$ be the set of \blacksquare Well-Formed Formulas that can be built up from \mathcal{S} by the five \blacksquare Formula-Building Operations. We define **extension** \bar{v} of v ,

$$\bar{v}: \bar{\mathcal{S}} \rightarrow \{F, T\},$$

as the function that satisfies the following conditions for any $\alpha, \beta \in \bar{\mathcal{S}}$:

(0) For any $A \in \mathcal{S}$, $\bar{v}(A) = v(A)$. (Thus \bar{v} is indeed an extension of v .)

$$(1) \bar{v}(\neg\alpha) = \begin{cases} T & \text{if } \bar{v}(\alpha) = F, \\ F & \text{otherwise.} \end{cases}$$

$$(2) \bar{v}(\alpha \wedge \beta) = \begin{cases} T & \text{if } \bar{v}(\alpha) = T \text{ and } \bar{v}(\beta) = T, \\ F & \text{otherwise.} \end{cases}$$

$$(3) \bar{v}(\alpha \vee \beta) = \begin{cases} T & \text{if } \bar{v}(\alpha) = T \text{ or } \bar{v}(\beta) = T \text{ (or both),} \\ F & \text{otherwise.} \end{cases}$$

$$(4) \bar{v}(\alpha \Rightarrow \beta) = \begin{cases} F & \text{if } \bar{v}(\alpha) = T \text{ and } \bar{v}(\beta) = F, \\ T & \text{otherwise.} \end{cases}$$

$$(5) \bar{v}(\alpha \Leftrightarrow \beta) = \begin{cases} T & \text{if } \bar{v}(\alpha) = \bar{v}(\beta), \\ F & \text{otherwise.} \end{cases}$$

We say that truth assignment v **satisfies** ϕ if and only if $\bar{v}(\phi) = T$.

\blacktriangledown - True

\blacktriangledown - False

R.8 \blacksquare Well-Formed Formula

A **well-formed formula** (wff) is an \blacksquare Expression that can be built up from the sentence symbols by applying some finite number of times the \blacksquare Formula-Building Operations.

 - Enderton.Logic.Chapter_1.Wff

Chapter 0

Useful Facts About Sets

0.1 Lemma 0A

Lemma 0A. Assume that $\langle x_1, \dots, x_m \rangle = \langle y_1, \dots, y_m, \dots, y_{m+k} \rangle$. Then $x_1 = \langle y_1, \dots, y_{k+1} \rangle$.

Proof. For natural number m , let $P(m)$ be the statement:

Induction Hypothesis (IH)

If $\langle x_1, \dots, x_m \rangle = \langle y_1, \dots, y_m, \dots, y_{m+k} \rangle$ then $x_1 = \langle y_1, \dots, y_{k+1} \rangle$.

We proceed by induction on m .

Base Case Suppose $\langle x_1 \rangle = \langle y_1, \dots, y_{1+k} \rangle$. By definition of an \blacksquare n -tuple, $\langle x_1 \rangle = x_1$. Thus $x_1 = \langle y_1, \dots, y_{k+1} \rangle$. Hence $P(1)$ holds true.

Inductive Step Suppose for $m \geq 1$ that $P(m)$ is true and assume

$$\langle x_1, \dots, x_{m+1} \rangle = \langle y_1, \dots, y_{m+1}, \dots, y_{m+1+k} \rangle. \quad (1)$$

By definition of an \blacksquare n -tuple, we can decompose (1) into the following two identities

$$\begin{aligned} x_{m+1} &= y_{m+1+k} \\ \langle x_1, \dots, x_m \rangle &= \langle y_1, \dots, y_{m+k} \rangle. \end{aligned}$$

By (IH), $P(m)$ implies $x_1 = \langle y_1, \dots, y_{k+1} \rangle$. Hence $P(m+1)$ holds true.

Conclusion By induction, $P(m)$ holds true for all $m \geq 1$. □

Chapter 1

Sentential Logic

1.1 The Language of Sentential Logic

1.1.1 Induction Principle

Theorem 1. *If S is a set of \ulcorner Well-Formed Formulas containing all the sentence symbols and closed under all five \ulcorner Formula-Building Operations, then S is the set of all wffs.*

 - [Enderton.Logic.Chapter_1.Wff.rec](#)

Proof. We note every well-formed formula can be characterized by a \ulcorner Construction Sequence. For natural number m , let $P(m)$ be the statement:

Induction Hypothesis (IH)

Every wff characterized by a construction sequence of length m is in S .

We proceed by strong induction on m .

Base Case Let ϕ denote a wff characterized by a construction sequence of length 1. Then it must be that ϕ is a single sentence symbol. By hypothesis, S contains all the sentence symbols. Thus $P(1)$ holds true.

Inductive Step Suppose $P(0), P(1), \dots, P(m)$ holds true and let ϕ denote a wff characterized by a construction sequence of length $m + 1$. By definition of a construction sequence, one of the following holds:

$$\phi \text{ is a sentence symbol} \tag{1.1}$$

$$\phi = \mathcal{E}_-(\epsilon_j) \text{ for some } j < m + 1 \tag{1.2}$$

$$\phi = \mathcal{E}_\square(\epsilon_j, \epsilon_k) \text{ for some } j < m + 1, k < m + 1 \tag{1.3}$$

where \square is one of the binary connectives $\wedge, \vee, \Rightarrow, \Leftrightarrow$. We consider each case in turn.

(1.1) By hypothesis, all sentence symbols are in S . Thus $\phi \in S$.

(1.2) Suppose $\phi = \mathcal{E}_{\neg}(\epsilon_j)$ for some $j < m + 1$. By (IH), ϵ_j is in S . By hypothesis, S is closed under \mathcal{E}_{\neg} . Thus $\phi \in S$.

(1.3) Suppose $\phi = \mathcal{E}_{\square}(\epsilon_j, \epsilon_k)$ for some $j < m + 1, k < m + 1$, By (IH), ϵ_j and ϵ_k is in S . By hypothesis, S is closed under \mathcal{E}_{\square} for all possible candidates of \square . Thus $\phi \in S$.

Subconclusion Since the above three cases are exhaustive, $P(m+1)$ holds.

Conclusion By strong induction, $P(m)$ holds true for all natural numbers $m \geq 1$. Since every well-formed formula is characterized by a construction sequence, the set of all wffs is a subset of S . Likewise, it obviously holds that S is a subset of all wffs. Thus S is precisely the set of all wffs. □

1.1.2 ☺ **Balanced Parentheses**

Lemma 2. All ¶ *Well-Formed Formulas have an equal number of left and right parentheses.*


Proof. Define

$$S = \{\phi \mid \phi \text{ is a wff with a balanced number of parentheses}\}.$$


We prove that (i) all the sentence symbols are members of S and (ii) S is closed under the five ¶ Formula-Building Operations. We then conclude with (iii) the proof of the theorem statement.

(i) By definition, well-formed formulas comprising a single sentence symbol do not have any parentheses. Thus all sentence symbols are members of S .

(ii) Let $\alpha, \beta \in S$. By definition, $\mathcal{E}_{\neg}(\alpha) = (\neg\alpha)$. Thus one additional left and right parenthesis is introduced. Since α is assumed to have an equal number of left and right parentheses, $\mathcal{E}_{\neg}(\alpha) \in S$. Likewise, $\mathcal{E}_{\square}(\alpha, \beta) = (\alpha \square \beta)$ where \square is one of the binary connectives $\wedge, \vee, \Rightarrow, \Leftrightarrow$. Again, an additional left and right parenthesis is introduced. Since α and β are assumed to have a balanced number of parentheses, $\mathcal{E}_{\square}(\alpha, \beta) \in S$. Hence S is closed under the five formula-building operations.

(iii) By (i) and (ii), the  **Induction Principle** implies S is the set of all wffs. Thus all well-formed formulas have an equal number of left and right parentheses. □


1.1.3 Parentheses Count

Lemma 3. *Let ϕ be a  Well-Formed Formula and c be the number of places at which a sentential connective symbol exists. Then there is $2c$ parentheses in ϕ .*

 - [Enderton.Logic.Chapter_1.paren_count_double_sentential_count](#)


Proof. Define

$$S = \{\phi \mid \phi \text{ is a wff with } 2c \text{ parentheses}\}.$$

We prove that (i) all the sentence symbols are members of S and (ii) S is closed under the five  Formula-Building Operations. We then conclude with (iii) the proof of the theorem statement.


(i) A sentence symbol, by itself, has no sentential connectives. Likewise, it has 0 parentheses. Thus S contains every sentence symbol.

(ii) Let $\alpha, \beta \in S$. By definition, $\mathcal{E}_-(\alpha) = (\neg\alpha)$. Then $\mathcal{E}_-(\alpha)$ introduces two additional parentheses and one additional sentential connective symbol. Thus $\mathcal{E}_-(\alpha) \in S$. Likewise, $\mathcal{E}_\square(\alpha, \beta) = (\alpha \square \beta)$ where \square is one of the binary connectives $\wedge, \vee, \Rightarrow, \Leftrightarrow$. $\mathcal{E}_\square(\alpha, \beta)$ also introduces two additional parentheses and one additional connective symbol. Thus $\mathcal{E}_\square(\alpha, \beta) \in S$. Hence S is closed under the five formula-building operations.

(iii) By (i) and (ii), the  **Induction Principle** implies S is the set of all wffs. Thus every wff has $2c$ parentheses in ϕ , where c denotes the number of places at which a sentential connective symbol exists. □



1.2 Truth Assignments

1.2.1 Theorem 12A

Theorem 12A. *For any  Truth Assignment v for a set \mathcal{S} there is a unique extension $\bar{v}: \mathcal{S} \rightarrow \{F, T\}$.*

Proof. TODO □

1.2.2 Compactness Theorem

Theorem 3. *Let Σ be an infinite set of  Well-Formed Formulas such that for any finite subset Σ_0 of Σ , there is a  Truth Assignment that satisfies every member of Σ_0 . Then there is a truth assignment that satisfies every member of Σ .*

Proof. TODO □

1.3 A Parsing Algorithm

1.3.1 Lemma 13A

Lemma 13A. *Every wff has the same number of left as right parentheses.*

Proof. Refer to  Balanced Parentheses. □

1.3.2 Lemma 13B

Lemma 13B. *Any proper initial segment of a wff contains an excess of left parentheses. Thus no proper initial segment of a wff can itself be a wff.*

Proof. TODO □

1.4 Exercises 1

1.4.1 Exercise 1.1.1

Give three sentences in English together with translations into our formal language. The sentences should be chosen so as to have an interesting structure, and the translations should each contain 15 or more symbols.

Answer. We begin first with the English sentences:

- (i) He can juggle beach balls, bowling pins, and hackysacks unless he is tired, in which case he can only juggle beach balls.
- (ii) If Lauren goes to the movies with Sam, he will watch Barbie and eat popcorn, but if Lauren does not, he will watch Oppenheimer and eat gummy worms.
- (iii) Trees produce oxygen if they are alive and well, able to pull nutrients from the earth, and receive ample water.

(i) We use the following translation: "To juggle beach balls" (B), "to juggle bowling pins" (P), "to juggle hackysacks" (H), and "he is tired" (T). This yields the following translation:

$$(B \wedge ((\neg T) \Rightarrow (P \wedge H))).$$

(ii) We use the following translation: "Lauren goes to the movies" (L), "Sam will watch Oppenheimer" (O), "Sam will watch "Barbie" (B), "Sam will eat popcorn" (P), and "Sam will eat gummy worms" (G). This yields the following translation:

$$(((L \wedge B) \wedge P) \vee (((\neg L) \wedge O) \wedge G)).$$

(iii) We use the following translation: "Trees produce oxygen" (O), "the tree is alive" (A), "the tree is well" (W), "can pull nutrients from the earth" (N), and "receives ample water" (R). This yields the following translation:

$$(O \iff (((A \wedge W) \wedge N) \wedge R)).$$

□

1.4.2 Exercise 1.1.2


Show that there are no wffs of length 2, 3, or 6, but that any other positive length is possible.

 - [Enderton.Logic.Chapter.1.exercise.1.1.2.i](#)

 - [Enderton.Logic.Chapter.1.exercise.1.1.2.ii](#)

Proof. Define

$$S = \{\phi \mid \phi \text{ is a wff and the length of } \phi \text{ is not } 2, 3, \text{ or } 6.\}.$$

We prove that (i) all the sentence symbols are members of S and (ii) S is closed under the five  Formula-Building Operations. We then conclude with (iii) the proof of the theorem statement.


(i) Sentence symbols, by definition, have length 1. Thus every sentence symbol is a member of S .

(ii) Define L to be the length function mapping arbitrary wff to its length. Let $\alpha, \beta \in S$. Then $L(\alpha)$ and $L(\beta)$ each evaluate to 1, 4, 5, or a value larger than 6.

By definition, $\mathcal{E}_{\neg}(\alpha) = (\neg\alpha)$. Thus $L(\mathcal{E}_{\neg}(\alpha)) = L(\alpha) + 3$. Enumerating through the possible values of $L(\alpha)$ shows $\mathcal{E}_{\neg}(\alpha) \in S$. Likewise, $\mathcal{E}_{\square}(\alpha, \beta) = (\alpha \square \beta)$ where \square is one of the binary connectives $\wedge, \vee, \Rightarrow, \Leftrightarrow$. Thus $L(\mathcal{E}_{\square}(\alpha, \beta)) =$


$L(\alpha) + L(\beta) + 3$. Again, enumerating through the possible values of $L(\alpha)$ and $L(\beta)$ shows $\mathcal{E}_\square(\alpha, \beta) \in S$.

Hence S is closed under the five formula-building operations.

(iii) By (i) and (ii), the  **Induction Principle** implies S is the set of all wffs. It remains to be shown that a wff of any positive length excluding 2, 3, and 6 are possible.

Let $\phi_1 = A_1$, $\phi_2 = (A_1 \wedge A_2)$, and $\phi_3 = ((A_1 \wedge A_2) \wedge A_3)$. Note these are wffs of lengths 1, 5, and 9 respectively. Then n repeated applications of \mathcal{E}_\square yields wffs of length $1 + 3n$, $5 + 3n$, and $9 + 3n$ respectively. But

$$\begin{aligned} &\{1 + 3n \mid n \in \mathbb{N}\}, \\ &\{5 + 3n \mid n \in \mathbb{N}\}, \text{ and} \\ &\{9 + 3n \mid n \in \mathbb{N}\} \end{aligned}$$

form a  Partition of set $\mathbb{N} - \{2, 3, 6\}$. Thus a wff of any other positive length besides 2, 3, and 6 is possible. □


1.4.3 Exercise 1.1.3

Let α be a wff; let c be the number of places at which binary connective symbols ($\wedge, \vee, \Rightarrow, \Leftrightarrow$) occur in α ; let s be the number of places at which sentence symbols occur in α . (For example, if α is $(A \Rightarrow (\neg A))$ then $c = 1$ and $s = 2$.) Show by using the induction principle that $s = c + 1$.

 - [Enderton.Logic.Chapter.1.exercise.1.1.3](#)

Proof. Define

$$S = \{\phi \mid \phi \text{ is a wff such that } s = c + 1\}. \quad (1.4)$$

We prove that (i) all the sentence symbols are members of S and (ii) S is closed under the five  Formula-Building Operations. We then conclude with (iii) the proof of the theorem statement.

(i) Let $\phi = A_n$ be an arbitrary sentence symbol. The number of places at which sentence symbols occur in ϕ is 1. The number of places at which binary connective symbols occur in ϕ is 0. Hence $\phi \in S$.

(ii) Let $\alpha, \beta \in S$. Denote the number of places at which sentence symbols occur in each as s_α and s_β respectively. Likewise, denote the number of places at which binary connective symbols occur as c_α and c_β .

By definition, $\mathcal{E}_\square(\alpha) = (\neg\alpha)$. The number of sentence and binary connective symbols in $\mathcal{E}_\square(\alpha)$ does not change. Thus $\mathcal{E}_\square(\alpha) \in S$. Likewise, $\mathcal{E}_\square(\alpha, \beta) = (\alpha \square \beta)$ where \square is one of the binary connectives $\wedge, \vee, \Rightarrow, \Leftrightarrow$. Therefore $\mathcal{E}_\square(\alpha, \beta)$ has

$s_\alpha + s_\beta$ sentence symbols and $c_\alpha + c_\beta + 1$ binary connective symbols. But (1.4) implies

$$\begin{aligned} s_\alpha + s_\beta &= (c_\alpha + 1) + (c_\beta + 1) \\ &= (c_\alpha + c_\beta + 1) + 1, \end{aligned}$$

meaning $\mathcal{E}_\square(\alpha, \beta) \in S$.

Hence S is closed under the five formula-building operations.

(iii) By (i) and (ii), the ☺ **Induction Principle** indicates S is the set of all wffs. □

1.4.4 ☺ **Exercise 1.1.4**

Assume we have a construction sequence ending in ϕ , where ϕ does not contain the symbol A_4 . Suppose we delete all the expressions in the construction sequence that contain A_4 . Show that the result is still a legal construction sequence.

Proof. Let S denote a ◼ Construction Sequence $\langle \epsilon_1, \dots, \epsilon_n \rangle$ such that $\epsilon_n = \phi$. Let $S' = \langle \epsilon_{i_1}, \dots, \epsilon_{i_m} \rangle$ denote the construction sequence resulting from deleting all expressions in S containing A_4 . Fix $1 \leq j \leq m$. Then there exists some $1 \leq k \leq n$ such that $\epsilon_{i_j} = \epsilon_k$. By definition of a construction sequence, there are three cases to consider:

Case 1 Suppose ϵ_k is a sentence symbol. Then ϵ_{i_j} is also sentence symbol.

Case 2 Suppose $\epsilon_k = \mathcal{E}_-(\epsilon_a)$ for some $a < k$. It must be that A_4 is not found in ϵ_a , else an immediate contradiction is raised. Therefore ϵ_a is a member of S' that precedes ϵ_{i_j} . Hence $\epsilon_{i_j} = \mathcal{E}_-(\epsilon_{i_a})$ for some $a < j$.

Case 3 Suppose $\epsilon_k = \mathcal{E}_\square(\epsilon_a, \epsilon_b)$ for some $a, b < k$ where \square is one of the binary connectives $\wedge, \vee, \Rightarrow, \Leftrightarrow$. It must be that A_4 is found in neither ϵ_a nor ϵ_b , else an immediate contradiction is raised. Therefore ϵ_a and ϵ_b is a member of S' , both of which precede ϵ_{i_j} . Hence $\epsilon_{i_j} = \mathcal{E}_\square(\epsilon_{i_a}, \epsilon_{i_b})$ for some $a, b < j$.

Conclusion Since the above cases are exhaustive and apply to an arbitrary member of S' , it must be that every member of S' is valid. Hence S' is still a legal construction sequence. □

1.4.5 ☑ **Exercise 1.1.5**

Suppose that α is a wff not containing the negation symbol \neg .

✔ **Exercise 1.1.5a**

Show that the length of α (i.e., the number of symbols in the string) is odd.
Suggestion: Apply induction to show that the length is of the form $4k + 1$.

🔗 - Enderton.Logic.Chapter_1.exercise.1.1.5a

Proof. Define L to be the length function mapping arbitrary \ulcorner Well-Formed Formula to its length and let

$$S = \{\phi \mid \phi \text{ is a wff containing } \neg \text{ or } \exists k \in \mathbb{N}, L(\phi) = 4k + 1\}. \quad (1.5)$$

We prove that (i) all the sentence symbols are members of S and (ii) S is closed under the five \ulcorner Formula-Building Operations. We then conclude with (iii) the proof of the theorem statement.

(i) Every sentence symbol has length 1 by definition. That is, every sentence symbol has length $(4)(0) + 1$. Hence S contains every sentence symbol.

(ii) Let $\alpha, \beta \in S$. Then there exists some k_α and k_β such that $L(\alpha) = 4k_\alpha + 1$ and $L(\beta) = 4k_\beta + 1$. Clearly S is closed under \mathcal{E}_\neg . Next consider $\mathcal{E}_\square(\alpha, \beta) = (\alpha \square \beta)$ where \square is one of the binary connectives $\wedge, \vee, \Rightarrow, \Leftrightarrow$. Then

$$\begin{aligned} L(\alpha, \beta) &= L(\alpha) + L(\beta) + 3 \\ &= (4k_\alpha + 1) + (4k_\beta + 1) + 3 \\ &= 4k_\alpha + 4k_\beta + 4 + 1 \\ &= 4(k_\alpha + k_\beta + 1) + 1. \end{aligned}$$

Therefore, there exists a $k \in \mathbb{N}$, namely $k = k_\alpha + k_\beta + 1$, such that $L(\mathcal{E}_\square(\alpha, \beta)) = 4k + 1$.

Hence S is closed under the five formula-building operations.

(iii) By (i) and (ii), the \odot Induction Principle indicates S is the set of all wffs. Thus all well-formed formulas not containing symbol \neg has length $4k + 1$ for some $k \in \mathbb{N}$. Therefore these well-formed formulas have odd length. \square

✔ **Exercise 1.1.5b**

Show that more than a quarter of the symbols are sentence symbols. *Suggestion:* Apply induction to show that the number of sentence symbols is of the form $k + 1$.

🔗 - Enderton.Logic.Chapter_1.exercise.1.1.5b

Proof. Let ϕ be a \blacksquare Well-Formed Formula. By \checkmark Exercise 1.1.3, the number of sentence symbols of ϕ is $k + 1$, where k is the number of places at which binary connective symbols occur in ϕ . By \checkmark Parentheses Count, the number of parentheses in ϕ is $2k$. Thus ϕ has length $(k + 1) + k + 2k = 4k + 1$. But

$$\frac{k + 1}{4k + 1} > \frac{k + 1}{4k + 4} = \frac{1}{4}.$$

Hence more than a quarter of the symbols of ϕ are sentence symbols. □

1.4.6 \checkmark Exercise 1.2.1

Show that neither of the following two formulas tautologically implies the other:

$$(A \Leftrightarrow (B \Leftrightarrow C)), \tag{1.6}$$

$$((A \wedge (B \wedge C)) \vee ((\neg A) \wedge ((\neg B) \wedge (\neg C)))). \tag{1.7}$$

Suggestion: Only two \blacksquare Truth Assignments are needed, not eight.

\mathfrak{P} - [Enderton.Logic.Chapter_1.exercise_1.2.1.i](#)

\mathfrak{P} - [Enderton.Logic.Chapter_1.exercise_1.2.1.ii](#)

Proof. First, suppose $A = T$, $B = F$, and $C = F$. Then (1.6) evaluates to T but (1.7) evaluates to F . Therefore (1.6) $\not\models$ (1.7).

Next, suppose $A = F$, $B = F$, and $C = F$. Then (1.7) evaluates to T but (1.6) evaluates to F . Therefore (1.7) $\not\models$ (1.6). □

1.4.7 \checkmark Exercise 1.2.2a

Is $((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$ a tautology?

\mathfrak{P} - [Enderton.Logic.Chapter_1.exercise_1.2.2a](#)

Proof. Yes. To prove, consider the following truth table:

$((P \Rightarrow Q) \Rightarrow P)$	\Rightarrow	Q	\Rightarrow	P	\Rightarrow	P
T	T	T	T	T	T	T
T	F	F	T	T	T	T
F	T	T	F	F	T	F
F	T	F	F	F	T	F

□

1.4.8 Exercise 1.2.2b

Define σ_k recursively as follows: $\sigma_0 = (P \Rightarrow Q)$ and $\sigma_{k+1} = (\sigma_k \Rightarrow P)$. For which values of k is σ_k a tautology? (Part (a) corresponds to $k = 2$.)

 - Enderton.Logic.Chapter_1.exercise.1_2_2b.i

 - Enderton.Logic.Chapter_1.exercise.1_2_2b.ii


 - Enderton.Logic.Chapter_1.exercise.1_2_2b.iii

Proof. We prove that σ_k is a tautology if and only if k is an even integer greater than zero. To do so, we show (i) that σ_k is a tautology for all even $k > 0$, (ii) σ_0 is not a tautology, and (iii) σ_k is not a tautology for all odd k .

(i) Let $P(k)$ be the predicate, " σ_k is a tautology." We prove $P(k)$ holds true for all even $k > 0$ via induction.

Base Case Let $k = 2$. By definition,

$$\sigma_2 = (((P \Rightarrow Q) \Rightarrow P) \Rightarrow P).$$

 Exercise 1.2.2a indicates σ_2 is a tautology. Hence $P(2)$ is true.

Inductive Step Suppose $P(k)$ holds for some even $k > 0$. By definition,

$$\sigma_{k+2} = ((\sigma_k \Rightarrow P) \Rightarrow P).$$

Consider the truth table of the above:

$((\sigma_k$	\Rightarrow	$P)$	\Rightarrow	$P)$
T	T	T	T	T
T	T	T	T	T
T	F	F	T	F
T	F	F	T	F

This shows σ_{k+2} is a tautology. Hence $P(k+2)$ is true.

Subconclusion By induction, $P(k)$ is true for all even $k > 0$.

(ii) By definition,

$$\sigma_0 = (P \Rightarrow Q).$$

This is clearly not a tautology since σ_0 evaluates to F when $P = T$ and $Q = F$.

(iii) Let $k > 0$ be an odd natural number. There are two cases to consider:

Case 1 Suppose $k = 1$. Then $\sigma_k = \sigma_1 = ((P \Rightarrow Q) \Rightarrow P)$. The following truth table shows σ_1 is not a tautology:

$((P \Rightarrow Q) \Rightarrow P)$	P	Q	\Rightarrow	P
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	T	F	F	F

Case 2 Suppose $k > 1$. Then $k - 1 > 0$ is an even number. By definition,

$$\sigma_k = (\sigma_{k-1} \Rightarrow P).$$

By (1.4.8), σ_{k-1} is a tautology. The following truth table shows σ_k is not:

$(\sigma_{k-1} \Rightarrow P)$	σ_{k-1}	\Rightarrow	P
T	T	T	T
T	T	F	T
T	F	F	F
T	F	T	F

□

1.4.9 ✓ Exercise 1.2.3a

Determine whether or not $((P \Rightarrow Q) \vee (Q \Rightarrow P))$ is a tautology.

🔗 - Enderton.Logic.Chapter.1.exercise.1.2.3a

Proof. Consider the following truth table:

$((P \Rightarrow Q) \vee (Q \Rightarrow P))$	P	Q	\vee	$(P \Rightarrow Q)$	$(Q \Rightarrow P)$
T	T	T	T	T	T
T	F	F	T	F	T
F	T	T	T	T	F
F	T	F	T	F	T

The above makes it immediately evident that $((P \Rightarrow Q) \vee (Q \Rightarrow P))$ is a tautology. □

1.4.10 ✓ Exercise 1.2.3b

Determine whether or not $((P \wedge Q) \Rightarrow R)$ tautologically implies $((P \Rightarrow R) \vee (Q \Rightarrow R))$.

🔗 - Enderton.Logic.Chapter.1.exercise.1.2.3b

Proof. Consider the following truth table:

P	Q	R	$((P \wedge Q) \Rightarrow R)$	$((P \Rightarrow R) \vee (Q \Rightarrow R))$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

The above makes it immediately evident that $((P \wedge Q) \Rightarrow R)$ tautologically implies $((P \Rightarrow R) \vee (Q \Rightarrow R))$. \square

1.4.11 Exercise 1.2.4

Show that the following hold:

- (a) $\Sigma; \alpha \models \beta$ iff $\Sigma \models (\alpha \Rightarrow \beta)$.
- (b) $\alpha \models \beta$ iff $\models (\alpha \Leftrightarrow \beta)$.

(Recall that $\Sigma; \alpha = \Sigma \cup \{\alpha\}$, the set Σ together with the one possibly new member α .)

Proof.

(a) We prove each direction of the biconditional.

(\Rightarrow) Assume $\Sigma; \alpha \models \beta$. Let v be a truth assignment for the sentence symbols in $\Sigma; \alpha$ and β . Then if v satisfies every member of Σ and α , it must also satisfy β . Denote $\bar{v}(\Sigma)$ as the proposition that v satisfies every member of Σ and consider the following truth table:

$\bar{v}(\Sigma)$	$\bar{v}(\alpha)$	$\bar{v}(\beta)$	$\bar{v}((\alpha \Rightarrow \beta))$
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	T

The red row denotes a contradiction: it is not possible for $\bar{v}(\Sigma)$ and $\bar{v}(\alpha)$ to be true but $\bar{v}(\beta)$ to be false. All remaining rows show that when $\bar{v}(\Sigma)$ is true, so is $\bar{v}((\alpha \Rightarrow \beta))$. Thus $\Sigma \models (\alpha \Rightarrow \beta)$.

(\Leftarrow) Assume $\Sigma \models (\alpha \Rightarrow \beta)$. Let v be a \blacksquare Truth Assignment for the sentence symbols in Σ , α , and β . Then if v satisfies every member of Σ , it must also satisfy $(\alpha \Rightarrow \beta)$. Denote $\bar{v}(\Sigma)$ as the proposition that v satisfies every member of Σ . By definition, $\bar{v}((\alpha \Rightarrow \beta)) = T$ if and only if $\bar{v}(\alpha) = F$ or $\bar{v}(\alpha)$ and $\bar{v}(\beta)$ are both true. Thus the only situation in which both $\bar{v}(\Sigma) = T$ and $\bar{v}(\alpha) = T$ corresponds to when $\bar{v}(\beta) = T$. Hence $\Sigma; \alpha \models \beta$.

(b) We prove each direction of the biconditional.

(\Rightarrow) Suppose $\alpha \models \beta$. Let v be a \blacksquare Truth Assignment for the sentence symbols in α and β . Consider the following truth table:

$(\alpha$	\Leftrightarrow	$\beta)$
T	T	T
T	F	F
F	F	T
F	T	F

The red rows indicate possibilities that cannot occur, for $\alpha \models \beta$ and $\beta \models \alpha$ by hypothesis. Of the remaining rows, $(\alpha \Leftrightarrow \beta)$ is true. Hence $\models (\alpha \Leftrightarrow \beta)$.

(\Leftarrow) Assume $\models (\alpha \Rightarrow \beta)$. Let v be a \blacksquare Truth Assignment for the sentence symbols in α and β . By definition, $\bar{v}((\alpha \Leftrightarrow \beta)) = T$ if and only if $\bar{v}(\alpha) = \bar{v}(\beta)$. Thus if $\bar{v}(\alpha)$ is true, so must $\bar{v}(\beta)$. That is, $\alpha \models \beta$. Likewise, if $\bar{v}(\beta)$ is true, so must $\bar{v}(\alpha)$. Therefore $\beta \models \alpha$. Hence $\alpha \models \beta$. □

1.4.12 Exercise 1.2.5

Prove or refute each of the following assertions:

- (a) If either $\Sigma \models \alpha$ or $\Sigma \models \beta$, then $\Sigma \models (\alpha \vee \beta)$.
- (b) If $\Sigma \models (\alpha \vee \beta)$, then either $\Sigma \models \alpha$ or $\Sigma \models \beta$.

- [Enderton.Logic.Chapter_1.exercise.1_2.5a](#)

- [Enderton.Logic.Chapter_1.exercise.1_2.5b](#)

Proof.

(a) WLOG, suppose $\Sigma \models \alpha$. That is, every truth assignment for sentence symbols found in Σ and α that satisfies every member of Σ also satisfies α . Let v be one of these truth assignments. Denote $\bar{v}(\Sigma)$ as the proposition that v

satisfies every member of Σ and consider the following truth table:

$\bar{v}(\Sigma)$	$\bar{v}(\alpha)$	$\bar{v}(\beta)$	$\bar{v}((\alpha \vee \beta))$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

The red rows indicate possibilities that cannot occur since $\Sigma \models \alpha$ by hypothesis. All remaining rows show that when $\bar{v}(\Sigma)$ is true, so is $\bar{v}((\alpha \vee \beta))$. Hence $\Sigma \models (\alpha \vee \beta)$.

(b) We proceed by counterexample. Suppose $\Sigma = \emptyset$. That is, assume $(\alpha \vee \beta)$ is a tautology, i.e. $\models (\alpha \vee \beta)$. Consider the following truth table:

α	\vee	β
T	T	T
T	T	F
F	T	T
F	F	F

The red row indicates an impossibility, since $(\alpha \vee \beta)$ should always be true by hypothesis. But this table also clearly demonstrates that $\not\models \alpha$ and $\not\models \beta$. Thus the conditional statement proposed must not be generally true. □

1.4.13 ☑ Exercise 1.2.6a

Show that if v_1 and v_2 are \blacksquare Truth Assignments which agree on all the sentence symbols in the wff α , then $\bar{v}_1(\alpha) = \bar{v}_2(\alpha)$. Use the ☑ Induction Principle.

Proof. Let σ map a \blacksquare Well-Formed Formula ϕ to the set of sentence symbols found in ϕ . Define

$$S = \{\phi \mid ((\sigma(\phi) = \sigma(\alpha)) \Rightarrow (\bar{v}_1(\phi) = \bar{v}_2(\phi)))\}. \quad (1.8)$$

We prove that (i) the set of sentence symbols is found in ϕ and (ii) S is closed under the five \blacksquare Formula-Building Operations. Afterward we show that (iii) our theorem statement holds.

(i) Let A_n denote an arbitrary sentence symbol. Suppose $\sigma(A_n) = \{A_n\} = \sigma(\alpha)$. But then $\bar{v}_1(A_n) = \bar{v}_2(A_n)$ since, by hypothesis, v_1 and v_2 agree on all the sentence symbols found in α . Hence S contains all the sentence symbols.

(ii) Let $\beta, \gamma \in S$. There are three cases to consider:

Case 1 Suppose $\sigma(\beta) \neq \sigma(\alpha)$. By definition, $\mathcal{E}_{\neg}(\beta) = (\neg\beta)$. Then clearly $\sigma(\mathcal{E}_{\neg}(\beta)) \neq \sigma(\alpha)$. Therefore $\mathcal{E}_{\neg}(\beta) \in S$. Likewise, $\mathcal{E}_{\square}(\beta, \gamma) = (\beta \square \gamma)$ where \square is one of the binary connectives $\wedge, \vee, \Rightarrow, \Leftrightarrow$. Again, it clearly follows that $\mathcal{E}_{\square}(\beta, \gamma) \neq \sigma(\alpha)$. Thus $\mathcal{E}_{\square}(\beta, \gamma) \in S$.

Case 2 Suppose $\sigma(\gamma) \neq \sigma(\alpha)$. This case mirrors [Case 1](#).

Case 3 Suppose $\sigma(\beta) = \sigma(\alpha) = \sigma(\alpha)$. By definition, $\mathcal{E}_{\neg}(\beta) = (\neg\beta)$. Then clearly $\sigma(\mathcal{E}_{\neg}(\beta)) = \sigma(\alpha)$. Since

$$\begin{aligned}\bar{v}_1((\neg\beta)) &= (\neg\bar{v}_1(\beta)) \\ &= (\neg\bar{v}_2(\beta)) \\ &= \bar{v}_2((\neg\beta)),\end{aligned}\tag{1.8}$$

it follows that $\mathcal{E}_{\neg}(\beta) \in S$.

Likewise, $\mathcal{E}_{\square}(\beta, \gamma) = (\beta \square \gamma)$ where \square is one of the binary connectives $\wedge, \vee, \Rightarrow, \Leftrightarrow$. Again, it clearly follows that $\mathcal{E}_{\square}(\beta, \gamma) \neq \sigma(\alpha)$. Since

$$\begin{aligned}\bar{v}_1((\alpha \square \beta)) &= (\bar{v}_1(\alpha) \square \bar{v}_1(\beta)) \\ &= (\bar{v}_2(\alpha) \square \bar{v}_2(\beta)) \\ &= \bar{v}_2((\alpha \square \beta)),\end{aligned}\tag{1.8}$$

it follows that $\mathcal{E}_{\square}(\beta, \gamma) \in S$.

Subconclusion The above three cases are exhaustive. Thus it follows S is closed under the five formula-building operations.

(iii) By (i) and (ii), the  **Induction Principle** indicates S is the set of all wffs. Since α is a well-formed formula, it follows $\alpha \in S$. Therefore

$$((\sigma(\alpha) = \sigma(\alpha)) \Rightarrow \bar{v}_1(\alpha) = \bar{v}_2(\alpha)).$$

The antecedent clearly holds true. Hence $\bar{v}_1(\alpha) = \bar{v}_2(\alpha)$ as expected. □

1.4.14 **Exercise 1.2.6b**

Let \mathcal{S} be a set of sentence symbols that includes those in Σ and τ (and possibly more). Show that $\Sigma \models \tau$ iff every truth assignment for \mathcal{S} which satisfies every member of Σ also satisfies τ . (This is an easy consequence of part (a). The point of part (b) is that we do not need to worry about getting the domain of a truth assignment *exactly* perfect, as long as it is big enough. For example, one option would be always to use truth assignments on the set of *all* sentence symbols. The drawback is that these are infinite objects, and there are a great many – uncountably many – of them.)

Proof. Let \mathcal{S} be a set of sentence symbols that includes those in Σ and τ (and possibly more). Let $\mathcal{S}' \subseteq \mathcal{S}$ be the set containing precisely the sentence symbols found in Σ and τ .

Let v be a truth assignment for \mathcal{S}' that satisfies every member of Σ and w be an arbitrary extension of v for \mathcal{S} . By construction, v and w agree on all the sentence symbols found in both Σ and τ . ☺ **Exercise 1.2.6a** then implies $\bar{v}(\tau) = \bar{w}(\tau)$ and $\bar{v}(\sigma) = \bar{w}(\sigma)$ for all $\sigma \in \Sigma$. Thus v satisfies every member of Σ if and only if w satisfies every member of Σ . Likewise, v satisfies τ if and only if w satisfies τ .

Hence, by definition of **☛ Tautological Implication**, $\Sigma \models \tau$ if and only if every truth assignment for the sentence symbols in \mathcal{S}' that satisfies every member of Σ also satisfies τ if and only if every truth assignment for the sentence symbols in \mathcal{S} that satisfies every member of Σ also satisfies τ . □

1.4.15 ☺ **Exercise 1.2.7**

You are in a land inhabited by people who either always tell the truth or always tell falsehoods. You come to a fork in the road and you need to know which fork leads to the capital. There is a local resident there, but he has time only to reply to one yes-or-no question. What one question should you ask so as to learn which fork to take? *Suggestion:* Make a table.

Proof. Consider the self-referential question, "Would you respond 'yes' to the question, 'Should I take the left road to get to the capital?'" Let I denote whether the inhabitant is truthful, and L denote whether the left road actually goes to the capital. We have R denote the answer given by the inhabitant in the following "truth table":

I	L	R
T	T	Yes
T	F	No
F	T	Yes
F	F	No

Regardless of the inhabitant's honesty, we receive the answer "Yes" if and only if the left road actually goes to the capital. □

1.4.16 ☺ **Exercise 1.2.8**

(Substitution) Consider a sequence $\alpha_1, \alpha_2, \dots$ of wffs. For each wff ϕ let ϕ^* be the result of replacing the sentence symbol A_n by α_n for each n .

☺ **Exercise 1.2.8a**

Let v be a truth assignment for the set of all sentence symbols; define u to be the truth assignment for which $u(A_n) = \bar{v}(\alpha_n)$. Show that $\bar{u}(\phi) = \bar{v}(\phi^*)$. Use

the \checkmark Induction Principle.

Proof. Let

$$S = \{\phi \mid \phi \text{ is a wff such that } \bar{u}(\phi) = \bar{v}(\phi^*)\}.$$

We prove that (i) S contains the set of all sentence symbols and (ii) S is closed under the five \blacksquare Formula-Building Operations. Afterward we prove that (iii) our theorem statement holds.

(i) Let $\phi = A_n$ be an arbitrary sentence symbol. By definition, $u(A_n) = \bar{v}(\alpha_n)$. Then

$$\bar{u}(\phi) = \bar{u}(A_n) = u(A_n) = \bar{v}(\alpha_n) = \bar{v}(\phi^*).$$

Hence every sentence symbol is in S .

(ii) Let $\beta, \gamma \in S$. That is, $\bar{u}(\beta) = \bar{v}(\beta^*)$ and $\bar{u}(\gamma) = \bar{v}(\gamma^*)$. By definition, $\mathcal{E}_-(\beta) = (\neg\beta)$. Therefore

$$\begin{aligned} \bar{u}(\mathcal{E}_-(\beta)) &= (\neg\bar{u}(\beta)) \\ &= (\neg\bar{v}(\beta^*)) \\ &= \bar{v}(\neg\beta^*) \\ &= \bar{v}((\neg\beta)^*) \\ &= \bar{v}(\mathcal{E}_-(\beta)^*). \end{aligned}$$

Likewise, $\mathcal{E}_\square(\beta, \gamma) = (\beta \square \gamma)$ where \square is one of the binary connectives $\wedge, \vee, \Rightarrow, \Leftrightarrow$. Therefore



$$\begin{aligned} \bar{u}(\mathcal{E}_\square(\beta, \gamma)) &= \bar{u}(\beta) \square \bar{u}(\gamma) \\ &= \bar{v}(\beta^*) \square \bar{v}(\gamma^*) \\ &= \bar{v}((\beta^* \square \gamma^*)) \\ &= \bar{v}((\beta \square \gamma)^*) \\ &= \bar{v}(\mathcal{E}_\square(\beta, \gamma)^*). \end{aligned}$$

Hence S is closed under the five formula-building operations.


(iii) By (i) and (ii), the \checkmark Induction Principle implies S is the set of all wffs. Thus for any well-formed formula ϕ , $\bar{u}(\phi) = \bar{v}(\phi^*)$. □

\checkmark Exercise 1.2.8b

Show that if ϕ is a tautology, then so is ϕ^* . (For example, one of our selected tautologies is $((A \wedge B) \Leftrightarrow (B \wedge A))$. From this we can conclude, by substitution, that $((\alpha \wedge \beta) \Leftrightarrow (\beta \wedge \alpha))$ is a tautology, for any wffs α and β .)

Proof. Suppose ϕ is a tautology and let S be the set of all sentence symbols. Let v be a truth assignment for S and define u to be the truth assignment for which $u(A_n) = \bar{v}(\alpha_n)$. By  **Exercise 1.2.8a**, $\bar{u}(\phi) = \bar{v}(\phi^*)$. Since ϕ is a tautology, $\bar{u}(\phi)$ is true meaning $\bar{v}(\phi^*)$ is also true. Since v is an arbitrary truth assignment, it follows that every truth assignment for S satisfies ϕ^* . By  **Exercise 1.2.6b**, $\models \phi^*$, i.e. ϕ^* is a tautology. \square

1.4.17 **Exercise 1.2.9**


(Duality) Let α be a wff whose only connective symbols are \wedge , \vee , and \neg . Let α^* be the result of interchanging \wedge and \vee and replacing each sentence symbol by its negation. Show that α^* is tautologically equivalent to $(\neg\alpha)$. Use the  **Induction Principle**.

 - not_and_de_morgan

 - not_or_de_morgan

Proof. Let

$$S = \{\alpha \mid \alpha \text{ is a wff containing a } \Rightarrow \text{ or } \\ \alpha \text{ is a wff containing a } \Leftrightarrow \text{ or } \\ \alpha^* \models \equiv (\neg\alpha)\}.$$

We prove that (i) S contains the set of all sentence symbols and (ii) S is closed under the five  Formula-Building Operations. Afterward we prove that (iii) our theorem statement holds.

(i) Let $\alpha = A_n$ be an arbitrary sentence symbol. By definition,

$$\alpha^* = A_n^* = (\neg A_n) = (\neg\alpha).$$

Hence every sentence symbol is in S .

(ii) Let $\alpha, \beta \in S$. Suppose α contains a \Rightarrow or \Leftrightarrow symbol. Then $\mathcal{E}_\neg(\alpha)$ also does. The same holds for β . Furthermore, if either α or β contains a \Rightarrow or \Leftrightarrow symbol, then so does $\mathcal{E}_\square(\alpha, \beta)$ where \square is one of the binary connectives \wedge , \vee , \Rightarrow , \Leftrightarrow . In any of these above cases, it is trivial to see each of the five-formula building operations take a wff from S and produce another wff in S .

Now, suppose neither α nor β contain a \Rightarrow or \Leftrightarrow symbol. Then it must be that $\alpha^* \models \equiv (\neg\alpha)$ and $\beta^* \models \equiv (\neg\beta)$. Consider first $\mathcal{E}_\neg(\alpha) = (\neg\alpha)$. By definition,

$$\mathcal{E}_\neg(\alpha)^* = (\neg\alpha^*) \models \equiv (\neg(\neg\alpha)) = (\neg(\mathcal{E}_\neg(\alpha))).$$

Therefore $\mathcal{E}_\neg(\alpha) \in S$.

Likewise, consider $\mathcal{E}_\square(\alpha, \beta)$ where \square is one of the binary connectives \wedge , \vee , \Rightarrow , \Leftrightarrow . It trivially follows that $\mathcal{E}_\Rightarrow(\alpha, \beta) \in S$ and $\mathcal{E}_\Leftrightarrow(\alpha, \beta) \in S$. We cover the remaining two cases in turn:

Case 1 Suppose $\square = \wedge$. Then

$$\begin{aligned}\mathcal{E}_\wedge(\alpha, \beta)^* &= (\alpha \wedge \beta)^* \\ &= (\alpha^* \vee \beta^*) \\ &\models ((\neg\alpha) \vee (\neg\beta)) \\ &\models \neg(\alpha \wedge \beta) \\ &= (\neg(\mathcal{E}_\wedge(\alpha, \beta))),\end{aligned}$$

where the last tautological equivalence follows from De Morgan's laws.

Case 2 Suppose $\square = \vee$. Then

$$\begin{aligned}\mathcal{E}_\vee(\alpha, \beta)^* &= (\alpha \vee \beta)^* \\ &= (\alpha^* \wedge \beta^*) \\ &\models ((\neg\alpha) \wedge (\neg\beta)) \\ &\models \neg(\alpha \vee \beta) \\ &= (\neg(\mathcal{E}_\vee(\alpha, \beta))),\end{aligned}$$

where the last tautological equivalence follows from De Morgan's laws.

Subconclusion The foregoing analysis shows that S is indeed closed under the five formula-building operations.

(iii) By (i) and (ii), the \odot **Induction Principle** implies S is the set of all wffs. Thus for any well-formed formula α whose only connective symbols are \wedge , \vee , and \neg , $\alpha^* \models (\neg\alpha)$. □

1.4.18 \odot Exercise 1.2.10

Say that a set Σ_1 of wffs is *equivalent* to a set Σ_2 of wffs iff for any wff α , we have $\Sigma_1 \models \alpha$ iff $\Sigma_2 \models \alpha$. A set Σ is *independent* iff no member of Σ is tautologically implied by the remaining members in Σ . Show that the following hold.

\odot Exercise 1.2.10a

A finite set of wffs has an independent equivalent subset.

Proof. For natural number n , let $P(n)$ be the statement:

Induction Hypothesis (IH)

A set of wffs $\#$ Equinumerous to n has an independent equivalent subset.

We proceed by induction on n .

Base Case Consider a finite set of wffs equinumerous to 0. This is simply the empty set. It is vacuously true that \emptyset is independent. Thus $\emptyset \subseteq \emptyset$ is an independent equivalent subset meaning $P(0)$ is true.

Inductive Step Suppose $P(n)$ holds true for some $n \geq 0$. That is, every finite set of wffs equinumerous to n has an independent equivalent subset. Consider now set Σ of wffs equinumerous to $n + 1$. There are two possibilities to consider:

Case 1 Suppose Σ is independent. Then $\Sigma \subseteq \Sigma$ is an independent equivalent subset.

Case 2 Suppose Σ is not independent. Then there exists some $\sigma \in \Sigma$ such that σ is tautologically implied by the remaining members of Σ . Let $\Sigma_1 = \Sigma - \{\sigma\}$. By (IH), Σ_1 has an independent equivalent subset Σ_2 .

Now let ϕ be an arbitrary wff. Then

$$\begin{aligned} \Sigma_2 \models \phi &\Rightarrow \Sigma_1 \models \phi && \text{def'n of equivalent} \\ &\Rightarrow \Sigma_1; \sigma \models \phi && \sigma \text{ is redundant} \\ &\Rightarrow \Sigma \models \phi. \end{aligned}$$

Likewise,

$$\begin{aligned} \Sigma \models \phi &\Rightarrow \Sigma_1; \sigma \models \phi \\ &\Rightarrow \Sigma_1 \models \phi && \sigma \text{ is redundant} \\ &\Rightarrow \Sigma_2 \models \phi. && \text{def'n of equivalent} \end{aligned}$$

Thus Σ_2 is an independent equivalent subset of Σ .

Subconclusion The above two cases are exhaustive. Hence $P(n+1)$ holds true.

Conclusion By induction, it follows $P(n)$ holds true for all $n \geq 0$. That is, every set of wffs equinumerous to a natural number has an independent equivalent subset. In other words, every finite set of wffs has an independent equivalent subset. □

☑ Exercise 1.2.10b


An infinite set need not have an independent equivalent subset.

Proof. Let

$$S = \{A_1 \wedge \cdots \wedge A_n \mid n \in \omega\}$$

be an infinite set of wffs. For the sake of contradiction, suppose S has an independent equivalent subset S' . There are two cases to consider:

Case 1 Suppose $S' = \emptyset$. Then it trivially follows S' is not equivalent to S , a contradiction.

Case 1 Suppose $S' \neq \emptyset$. By the  **Well Ordering of ω** , there exists a least $n \in \mathbb{N}$ such that $\phi = A_1 \wedge \dots \wedge A_n$ is in S' . It cannot be that another element of S' exists since such an element would tautologically imply ϕ , contradicting independence. Thus $S' = \{\phi\}$. But $\{\phi\}$ cannot be equivalent to S since it has no information about sentence symbol e.g. A_{n+1} , another contradiction.

Conclusion The above two cases are exhaustive and both yield contradictions. It must be that S does not have an independent equivalent subset. □

1.4.19 Exercise 1.2.11

Show that a truth assignment v satisfies the wff

$$(\dots(A_1 \Leftrightarrow A_2) \Leftrightarrow \dots \Leftrightarrow A_n)$$

iff $v(A_i) = F$ for an even number of i 's, $1 \leq i \leq n$. (By the associative law for \Leftrightarrow , the placement of the parentheses is not crucial.)

Proof. Define σ_n recursively as follows: $\sigma_0 = (A_1 \Leftrightarrow A_2)$ and $\sigma_{n+1} = (\sigma_n \Leftrightarrow A_{n+3})$. For natural number n , let $P(n)$ be the statement:

Induction Hypothesis (IH)

Truth assignment v satisfies σ_n if and only if $v(A_i) = F$ for an even number of i 's, $1 \leq i \leq n + 2$.

We proceed by induction on n .

Base Case Let $n = 0$. Then $\sigma_n = \sigma_0 = (A_1 \Leftrightarrow A_2)$. We proceed by truth table:

A_1	\Leftrightarrow	A_2
T	T	T
T	F	F
F	F	T
F	T	F

Here we see $A_1 \Leftrightarrow A_2$ is true if and only if both A_1 and A_2 are true or neither A_1 nor A_2 are true. Thus $P(0)$ holds true.

Inductive Step Suppose $P(n)$ holds true for some $n \geq 0$. Consider now

$$\sigma_{n+1} = (\sigma_n \Leftrightarrow A_{n+3}).$$

Let v be a truth assignment for A_1, \dots, A_{n+3} . There are two cases to consider:

Case 1 Suppose $v(A_i) = F$ for an even number of i 's, $1 \leq i \leq n + 2$. By (IH), v satisfies σ_n . We now have the following truth table:

$(\sigma_n \Leftrightarrow A_{n+3})$
$T \quad T \quad T$
$T \quad F \quad F$

In this case, it follows v satisfies σ_{n+1} if and only if $v(A_i) = F$ for an even number of i 's, $1 \leq i \leq n + 3$.

Case 2 Suppose $v(A_i) = F$ for an odd number of i 's, $1 \leq i \leq n + 2$. By (IH), v does not satisfy σ_n . We now have the following truth table:

$(\sigma_n \Leftrightarrow A_{n+3})$
$F \quad F \quad T$
$F \quad T \quad F$

In this case, it follows v satisfies σ_{n+1} if and only if $v(A_i) = F$ for an even number of i 's, $1 \leq i \leq n + 3$.

Subconclusion The above two cases are exhaustive. Hence $P(n+1)$ holds true.

Conclusion By induction, it follows $P(n)$ holds true for all $n \geq 0$. That is, truth assignment v satisfies

$$(\cdots(A_1 \Leftrightarrow A_2) \Leftrightarrow \cdots \Leftrightarrow A_n)$$

if and only if $v(A_i) = F$ for an even number of i 's, $1 \leq i \leq n$. □

1.4.20 ☑ Exercise 1.2.12

There are three suspects for a murder: Adams, Brown, and Clark. Adams says "I didn't do it. The victim was an old acquaintance of Brown's. But Clark hated him." Brown states "I didn't do it. I didn't even know the guy. Besides I was out of town all that week." Clark says "I didn't do it. I saw both Adams and Brown downtown with the victim that day; one of them must have done it." Assume that the two innocent men are telling the truth, but that the guilty man might not be. Who did it?

Proof. It must be that Brown is the guilty one. Adam claims the victim was an old acquaintance of Brown's. Clark claims Brown was downtown with the victim that day. Brown's testimony conflicts with both of these statements. □

1.4.21 ☑ Exercise 1.2.13

An advertisement for a tennis magazine states, "If I'm not playing tennis, I'm watching tennis. And if I'm not watching tennis, I'm reading about tennis." We can assume that the speaker cannot do more than one of these activities at a time. What is the speaker doing? (Translate the given sentences into our formal language; consider the possible truth assignments.)

Proof. Let P denote playing tennis, W denote watching tennis, and R denote reading about tennis. These statements can be translated as:

- (a) $\neg P \Rightarrow W$.
- (b) $\neg W \Rightarrow R$.

Thus either the speaker is playing tennis, or, if not, he is watching tennis. Since we assume the speaker cannot do more than one of these activities at a time, reading is never a possibility. \square

1.4.22 ! Exercise 1.2.14

Let \mathcal{S} be the set of all sentence symbols, and assume that $v: \mathcal{S} \rightarrow \{F, T\}$ is a truth assignment. Show there is *at most* one extension \bar{v} meeting conditions 0-5 listed at the beginning of this section. (Suppose that \bar{v}_1 and \bar{v}_2 are both such extensions. Use the ☑ Induction Principle to show that $\bar{v}_1 = \bar{v}_2$.)

Proof. The conditions 0-5 can be found at ¶ Truth Assignment. TODO \square

1.4.23 ☑ Exercise 1.2.15


Of the following three formulas, which tautologically implies which?

- (a) $(A \Leftrightarrow B)$
- (b) $(\neg((A \Rightarrow B) \Rightarrow (\neg(B \Rightarrow A))))$
- (c) $(((\neg A) \vee B) \wedge (A \vee (\neg B)))$

📖 - [Enderton.Logic.Chapter.1.exercise.1.2.15.i](#)


📖 - [Enderton.Logic.Chapter.1.exercise.1.2.15.ii](#)

Proof. All three are tautologically equivalent. We prove that (i) (a) is tautologically equivalent to (b) and (ii) (a) is tautologically equivalent to (c). It then immediately follows that (b) is tautologically equivalent to (c).

(i) By  Exercise 1.2.4, $(a) \models (b)$ if and only if $\models ((a) \Leftrightarrow (b))$. We now construct the corresponding truth table:

$(A \Leftrightarrow B)$	\Leftrightarrow	$(\neg ((A \Rightarrow B) \Rightarrow (\neg (B \Rightarrow A))))$
T	T	T
T	F	F
F	F	T
F	T	T

Therefore (a) and (b) are tautologically equivalent.

(ii) By  Exercise 1.2.4, $(a) \models (c)$ if and only if $\models ((a) \Leftrightarrow (c))$. We now construct the corresponding truth table:

$(A \Leftrightarrow B)$	\Leftrightarrow	$((\neg A) \vee B) \wedge (A \vee \neg B)$
T	T	T
T	F	F
F	F	T
F	T	T

□

1.4.24 Exercise 1.3.1

Rewrite the tautologies in the "selected list" at the end of Section 1.2, but using the conventions of the present section to minimize the number of parentheses.

Answer. TODO

□

1.4.25 Exercise 1.3.2

Give an example of wffs α and β and expressions γ and δ such that $(\alpha \wedge \beta) = (\gamma \wedge \delta)$ but $\alpha \neq \gamma$.

Answer. TODO

□

1.4.26 Exercise 1.3.3

Carry out the argument for  Lemma 13B for the case of the operation \mathcal{E}_- .

Answer. TODO

□

1.4.27 Exercise 1.3.4

Suppose that we modify our definition of wff by omitting all *right* parentheses. Thus instead of

$$((A \wedge (\neg B)) \Rightarrow (C \vee D))$$

we use

$$((A \wedge (\neg B \Rightarrow (C \wedge D.$$

Show that we still have unique readability (i.e., each wff still has only one possible decomposition). *Suggestion:* These expressions have the same number of parentheses as connective symbols.

Answer. TODO □

1.4.28 Exercise 1.3.5

The English language has a tendency to use two-part connectives: "both ... and ..." "either ... or ..." "if ..., then ..." How does this affect unique readability in English?

Answer. TODO □

1.4.29 Exercise 1.3.6

We have given an algorithm for analyzing a wff by constructing its tree from the top down. There are also ways of constructing the tree from the bottom up. This can be done by looking through the formula for innermost pairs of parentheses. Give a complete description of an algorithm of this sort.

Answer. TODO □

1.4.30 Exercise 1.3.7

Suppose that left and right parentheses are indistinguishable. Thus, instead of $\alpha \vee (\beta \wedge \gamma)$ we have $|\alpha \vee |\beta \wedge \gamma||$. Do formulas still have unique decomposition?

Answer. TODO □