Sequences

Contents

1 Summations **1**
1.1 Arithmetic Series 1
1.2 Geometric Series 2

1 Summations

1.1 Arithmetic Series

Let $(a_i)_{i \geqslant 0}$ be an arithmetic sequence with common difference d. Then for some $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} a_i = \frac{(n+1)(a_0 + a_n)}{2}. \tag{1}$$

- Real.Arithmetic.sum_recursive_closed

Proof. Let $(a_i)_{i \geqslant 0}$ be an arithmetic sequence with common difference d. By definition, for all $k \in \mathbb{N}$,

$$a_k = (a_0 + kd). \tag{2}$$

Define predicate $P(n)$ as "identity (1) holds for value n." We use induction to prove $P(n)$ holds for all $n \geqslant 0$.

Base Case Let $k = 0$. Then

$$\sum_{i=0}^{k} a_i = a_0 = \frac{2a_0}{2} = \frac{(k+1)(a_0 + a_k)}{2}.$$

Therefore $P(0)$ holds.

Induction Step Assume induction hypothesis $P(k)$ holds for some $k \geqslant 0$. Then

$$\begin{aligned}
\sum_{i=0}^{k+1} a_i &= \sum_{i=0}^{k} a_i + a_{k+1} \\
&= \frac{(k+1)(a_0 + a_k)}{2} + a_{k+1} && \text{induction hypothesis} \\
&= \frac{(k+1)(a_0 + (a_0 + kd))}{2} + (a_0 + (k+1)d) && (2) \\
&= \frac{(k+1)(2a_0 + kd)}{2} + (a_0 + (k+1)d) \\
&= \frac{(k+1)(2a_0 + kd) + 2a_0 + 2(k+1)d}{2} \\
&= \frac{2ka_0 + k^2 d + 4a_0 + kd + 2kd + 2d}{2} \\
&= \frac{(k+2)(2a_0 + kd + d)}{2} \\
&= \frac{(k+2)(a_0 + a_0 + (k+1)d)}{2} \\
&= \frac{(k+2)(a_0 + a_{k+1})}{2} && (2) \\
&= \frac{((k+1)+1)(a_0 + a_{k+1})}{2}.
\end{aligned}$$

Thus $P(k)$ implies $P(k+1)$ holds true.

Conclusion By mathematical induction, it follows for all $n \geqslant 0$, $P(n)$ is true. □

1.2 Geometric Series

Let $(a_i)_{i \geqslant 0}$ be a geometric sequence with common ratio $r \neq 1$. Then for some $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} a_i = \frac{a_0(1 - r^{n+1})}{1 - r}. \tag{3}$$

- Real.Geometric.sum_recursive_closed

Proof. Let $(a_i)_{i \geqslant 0}$ be a geometric sequence with common ratio $r \neq 1$. By definition, for all $k \in \mathbb{N}$,

$$a_k = a_0 r^k. \tag{4}$$

Define predicate $P(n)$ as "identity (3) holds for value n." We use induction to prove $P(n)$ holds for all $n \geqslant 0$.

Base Case Let $k = 0$. Then

$$\sum_{i=0}^{k} a_i = a_0 = \frac{a_0(1-r)}{1-r} = \frac{a_0(1-r^{k+1})}{1-r}$$

Therefore $P(0)$ holds.

Induction Step Assume induction hypothesis $P(k)$ holds for some $k \geqslant 0$. Then

$$\begin{aligned}
\sum_{i=0}^{k+1} a_i &= \sum_{i=0}^{k} a_i + a_{k+1} \\
&= \frac{a_0(1-r^{k+1})}{1-r} + a_{k+1} && \text{induction hypothesis} \\
&= \frac{a_0(1-r^{k+1})}{1-r} + a_0 r^{k+1} && (4) \\
&= \frac{a_0(1-r^{k+1}) + a_0 r^{k+1}(1-r)}{1-r} \\
&= \frac{a_0(1-r^{k+1} + r^{k+1}(1-r))}{1-r} \\
&= \frac{a_0(1-r^{k+1} + r^{k+1} - r^{k+2})}{1-r} \\
&= \frac{a_0(1-r^{k+2})}{1-r} \\
&= \frac{a_0(1-r^{(k+1)+1})}{1-r}.
\end{aligned}$$

Thus $P(k)$ implies $P(k+1)$ holds true.

Conclusion By mathematical induction, it follows for all $n \geqslant 0$, $P(n)$ is true.

□